mechanics, Pt. II, Hydrodynamics, 1935, pp. 219–232; Milne-Thomson, L. M., *Theoretical Hydrodynamics*, 1938, pp. 323–348.

[4] Lagally, M., *Math. Zeits.*, **10**, 231–239 (1921).

[5] Masotti, A., *Atti Pontif. Accad. Sci. Nuovi Lincei*, **84**, 209–216, 235–245, 464–467, 468–473, 623–631 (1931). Also, *Seminario Matematico e Fisico di Milano*, **6**, 3–53 (1932).

[6] The function called by Lagally the Kirchhoff's path function shall be called in this paper the "Kirchhoff-Routh function." The study of the function called by him the Routh's stream function is not of much importance, because it is merely a special application of the other (cf. equation (6.1)).

[7] For the definition of $O(\ )$ and $o(\ )$, cf. Titchmarsh, E. C., *Theory of Functions*, 1932, p. 1, Oxford.

[8] Koebe, P., *Acta Math.*, **41**, 306–344 (1918). Note that our function in case (*b*) is the function with two singularities, one at P_0, the other at infinity.

[9] Cf. Kellogg, O. D., *Foundations of Potential Theory*, 1929, pp. 238–240, Berlin. Note that no assumption is made regarding the nature of the boundaries $C_0, C_1, \ldots, C_k$.

ON THE MOTION OF VORTICES IN TWO DIMENSIONS—II SOME FURTHER INVESTIGATIONS ON THE KIRCHHOFF-ROUTH FUNCTION

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5. *Conformal Transformation.*—We shall now investigate the behavior of the Kirchhoff-Routh function (whose existence we have established in the preceding article) under a conformal transformation of fluid motion.

THEOREM II (*Generalized Routh's theorem*).—*Under a conformal transformation*

$$\tilde{z} = f(z) \tag{5.1}$$

which derives the motion in the $\tilde{z}$-plane from that in the z-plane, the Kirchhoff-Routh function for the new motion is given by

$$\tilde{W} = W + \sum_{i=1}^{n} \frac{\kappa_i^2}{4\pi} \log \left|\frac{dz}{d\tilde{z}}\right|_{P_i} \tag{5.2}$$

Proof. If $F(z)$ is the complex stream function in the z-plane, we have (*cf.*(2.3))

$$-u_i + iv_i = \lim_{P \to P_i} \frac{d}{dz}\left\{F(z) - \frac{i\kappa_i}{2\pi}\log(z - z_i)\right\}. \tag{5.3}$$

We mark every quantity in the $\tilde{z}$-plane with a curl. The complex stream function for the new motion is then

$$\tilde{F}(\tilde{z}) = F(z), \tag{5.4}$$

by definition of conformal transformation. From this it follows that $\tilde{\kappa}_i = \kappa_i$, $(i = 1, 2, \ldots, n)$. It can be verified that, by (5.3) and (5.4),

$$-\tilde{u}_i + i\tilde{v}_i = (-u_i + iv_i)\frac{dz_i}{d\tilde{z}_i} + \frac{i\kappa_i}{4\pi}\frac{d}{d\tilde{z}_i}\log\left(\frac{dz_i}{d\tilde{z}_i}\right). \tag{5.5}$$

Multiplying (5.5) by δz_i and taking imaginary parts, we have

$$\tilde{v}_i\delta\tilde{x}_i - \tilde{u}_i\delta\tilde{y}_i = v_i\delta x_i - u_i\delta y_i + \frac{\kappa_i}{4\pi}\delta\log\left|\frac{dz}{d\tilde{z}}\right|_{P_i}. \tag{5.6}$$

Multiplying (5.6) by κ_i and summing for the index i, we have, by (4.3),

$$\delta\tilde{W} = \delta W + \delta\sum_{i=1}^{n}\frac{\kappa_i^2}{4\pi}\log\left|\frac{dz}{d\tilde{z}}\right|_{P_i}. \tag{5.7}$$

Equation (5.2) then follows at once (up to an additive constant).

6. *Discussion.*—(i) The above results hold when the solid boundaries are moving, for this affects the stream function ψ_0 alone. The Green function G, of course, depends upon the instantaneous configuration of the solid boundaries. Furthermore, the results hold also when the function ψ_0 has *fixed* singularities.

(ii) The theory gives explicitly the Kirchhoff-Routh function W by the formula (4.4) when the ordinary stream function (4.2) is known, for the functions ψ_0, G and g can then be written down at once. In actual applications, we must be careful to see that ψ is actually in the form (4.2), with ψ_0 and G satisfying required conditions.

(iii) *Routh's stream function and Routh's stream.*—By (4.3), the motion of the ith vortex may be derived from a *Routh's theorem function*

$$\chi_{(i)} = \frac{W}{\kappa_i} \tag{6.1}$$

by the formulas

$$u_i = -\frac{\partial\chi_{(i)}}{\partial y_i},\ v_i = \frac{\partial\chi_{(i)}}{\partial x_i}, \tag{6.2}$$

just as the fluid velocity is derived from the ordinary stream function ψ. By (4.4) and (5.2), we see that

$$\chi_{(i)} = \psi_0(x_i, y_i) + \sum_{j \neq i}\kappa_j G(x_i, y_i;\ x_j, y_j) + \frac{\kappa_i}{2}g(x_i, y_i;\ x_i, y_i) + h_i, \tag{6.3}$$

and transforms according to the law

$$\chi_{(i)} = \chi_{(i)} + \frac{\kappa_i}{4\pi}\log\left|\frac{dz}{d\tilde{z}}\right|_{P_i} + k_i, \tag{6.4}$$

under the transformation (5.1). In these equations, h_i and k_i are independent of x_i and y_i, and are therefore unimportant so far as the motion of the ith vortex is concerned. Equation (6.4) is *Routh's theorem* generalized to the case of a number of vortices.

Routh's original special case.—For a single vortex κ_0 at the point $P_0(x_0, y_0)$, the equations corresponding to (6.3) and (6.4) are

$$\chi = \frac{W}{\kappa_0} = \psi_0(x_0, y_0) + \frac{\kappa_0}{2} g(x_0, y_0;\ x_0, y_0) \tag{6.5}$$

and

$$\tilde{\chi} = \chi + \frac{\kappa_0}{4\pi} \log \left| \frac{dz}{d\tilde{z}} \right|_{P_0}. \tag{6.6}$$

Equation (6.6) is *Routh's theorem* in its original form. In this case, the path of the vortex is given by

$$\chi = \text{const.}, \tag{6.7}$$

if the boundaries are fixed.

(iv) *Kirchhoff's original special case.*—If all the solid boundaries are absent, we have $g \equiv 0$. If, furthermore, there is no motion beyond that *due* to the vortices themselves, (4.4) reduces to the simple result

$$W = \frac{1}{2\pi} \sum_{\substack{i, j = 1 \\ (i > j)}}^{n} \kappa_i \kappa_j \log r_{ij}, \quad r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}, \tag{6.8}$$

first derived directly by Kirchhoff.[1]

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[1] Kirchhoff, G., *Vorlesungen über Mathematische Physik, Mechanik*, p. 225 ff. See also Lamb, H., *Hydrodynamics*, 1932, p. 230.